Exercise 3

Evaluate $(\nabla \cdot \mathbf{v})$, $\nabla \mathbf{v}$, and $[\nabla \cdot \mathbf{vv}]$ for the four fields in Exercise 2.

Solution

$\nabla \cdot \mathbf{v}$ is the divergence of the vector field $\mathbf{v}$, and it is computed by

$$\begin{aligned}\nabla \cdot \mathbf{v} &= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \cdot \left(\sum_{j=1}^{3} \boldsymbol{\delta}_j v_j\right) = \sum_{i=1}^{3}\sum_{j=1}^{3} (\boldsymbol{\delta}_i \cdot \boldsymbol{\delta}_j) \frac{\partial v_j}{\partial x_i} = \sum_{i=1}^{3}\sum_{j=1}^{3} \delta_{ij} \frac{\partial v_j}{\partial x_i} = \sum_{i=1}^{3} \frac{\partial v_i}{\partial x_i} \\ &= \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}.\end{aligned}$$

$\nabla \mathbf{v}$ is the gradient of the vector field $\mathbf{v}$, and it is computed by

$$\begin{aligned}\nabla \mathbf{v} &= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \left(\sum_{j=1}^{3} \boldsymbol{\delta}_j v_j\right) = \sum_{i=1}^{3}\sum_{j=1}^{3} \boldsymbol{\delta}_i \boldsymbol{\delta}_j \frac{\partial v_j}{\partial x_i} = \sum_{i=1}^{3} \left(\boldsymbol{\delta}_i \boldsymbol{\delta}_1 \frac{\partial v_1}{\partial x_i} + \boldsymbol{\delta}_i \boldsymbol{\delta}_2 \frac{\partial v_2}{\partial x_i} + \boldsymbol{\delta}_i \boldsymbol{\delta}_3 \frac{\partial v_3}{\partial x_i}\right) \\ &= \boldsymbol{\delta}_1 \boldsymbol{\delta}_1 \frac{\partial v_1}{\partial x_1} + \boldsymbol{\delta}_1 \boldsymbol{\delta}_2 \frac{\partial v_2}{\partial x_1} + \boldsymbol{\delta}_1 \boldsymbol{\delta}_3 \frac{\partial v_3}{\partial x_1} + \boldsymbol{\delta}_2 \boldsymbol{\delta}_1 \frac{\partial v_1}{\partial x_2} + \boldsymbol{\delta}_2 \boldsymbol{\delta}_2 \frac{\partial v_2}{\partial x_2} + \boldsymbol{\delta}_2 \boldsymbol{\delta}_3 \frac{\partial v_3}{\partial x_2} \\ &\qquad + \boldsymbol{\delta}_3 \boldsymbol{\delta}_1 \frac{\partial v_1}{\partial x_3} + \boldsymbol{\delta}_3 \boldsymbol{\delta}_2 \frac{\partial v_2}{\partial x_3} + \boldsymbol{\delta}_3 \boldsymbol{\delta}_3 \frac{\partial v_3}{\partial x_3}.\end{aligned}$$

$\nabla \cdot \mathbf{vv}$ is the divergence of the dyadic product $\mathbf{vv}$, and it is computed by

$$\begin{aligned}\nabla \cdot \mathbf{vv} &= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \cdot \left(\sum_{j=1}^{3} \boldsymbol{\delta}_j v_j\right) \left(\sum_{k=1}^{3} \boldsymbol{\delta}_k v_k\right) = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} (\boldsymbol{\delta}_i \cdot \boldsymbol{\delta}_j) \boldsymbol{\delta}_k \frac{\partial}{\partial x_i}(v_j v_k) \\ &= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \delta_{ij} \boldsymbol{\delta}_k \frac{\partial}{\partial x_i}(v_j v_k) = \sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k \frac{\partial}{\partial x_j}(v_j v_k) \\ &= \sum_{k=1}^{3} \left[\boldsymbol{\delta}_k \frac{\partial}{\partial x_1}(v_1 v_k) + \boldsymbol{\delta}_k \frac{\partial}{\partial x_2}(v_2 v_k) + \boldsymbol{\delta}_k \frac{\partial}{\partial x_3}(v_3 v_k)\right] \\ &= \sum_{k=1}^{3} \boldsymbol{\delta}_k \left[\frac{\partial}{\partial x_1}(v_1 v_k) + \frac{\partial}{\partial x_2}(v_2 v_k) + \frac{\partial}{\partial x_3}(v_3 v_k)\right] \\ &= \boldsymbol{\delta}_1 \left[\frac{\partial}{\partial x_1}(v_1 v_1) + \frac{\partial}{\partial x_2}(v_2 v_1) + \frac{\partial}{\partial x_3}(v_3 v_1)\right] + \boldsymbol{\delta}_2 \left[\frac{\partial}{\partial x_1}(v_1 v_2) + \frac{\partial}{\partial x_2}(v_2 v_2) + \frac{\partial}{\partial x_3}(v_3 v_2)\right] \\ &\qquad + \boldsymbol{\delta}_3 \left[\frac{\partial}{\partial x_1}(v_1 v_3) + \frac{\partial}{\partial x_2}(v_2 v_3) + \frac{\partial}{\partial x_3}(v_3 v_3)\right].\end{aligned}$$

Part (a)

The first vector field has components $v_x = by$, $v_y = 0$, and $v_z = 0$. Using the formulas, we have

$$\begin{aligned}
\nabla \cdot \mathbf{v} &= \frac{\partial}{\partial x}(by) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(0) = 0 \\
\nabla \mathbf{v} &= \boldsymbol{\delta}_x \boldsymbol{\delta}_x \frac{\partial}{\partial x}(by) + \boldsymbol{\delta}_x \boldsymbol{\delta}_y \frac{\partial}{\partial x}(0) + \boldsymbol{\delta}_x \boldsymbol{\delta}_z \frac{\partial}{\partial x}(0) + \boldsymbol{\delta}_y \boldsymbol{\delta}_x \frac{\partial}{\partial y}(by) + \boldsymbol{\delta}_y \boldsymbol{\delta}_y \frac{\partial}{\partial y}(0) + \boldsymbol{\delta}_y \boldsymbol{\delta}_z \frac{\partial}{\partial y}(0) \\
&\quad + \boldsymbol{\delta}_z \boldsymbol{\delta}_x \frac{\partial}{\partial z}(by) + \boldsymbol{\delta}_z \boldsymbol{\delta}_y \frac{\partial}{\partial z}(0) + \boldsymbol{\delta}_z \boldsymbol{\delta}_z \frac{\partial}{\partial z}(0) \\
&= b \boldsymbol{\delta}_y \boldsymbol{\delta}_x \\
\nabla \cdot \mathbf{v}\mathbf{v} &= \boldsymbol{\delta}_x \left[\frac{\partial}{\partial x}(b^2 y^2) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(0)\right] + \boldsymbol{\delta}_y \left[\frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(0)\right] \\
&\quad + \boldsymbol{\delta}_z \left[\frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(0)\right] \\
&= \mathbf{0}.
\end{aligned}$$

Part (b)

The second vector field has components $v_x = bx$, $v_y = 0$, and $v_z = 0$. Using the formulas, we have

$$\begin{aligned}
\nabla \cdot \mathbf{v} &= \frac{\partial}{\partial x}(bx) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(0) = b \\
\nabla \mathbf{v} &= \boldsymbol{\delta}_x \boldsymbol{\delta}_x \frac{\partial}{\partial x}(bx) + \boldsymbol{\delta}_x \boldsymbol{\delta}_y \frac{\partial}{\partial x}(0) + \boldsymbol{\delta}_x \boldsymbol{\delta}_z \frac{\partial}{\partial x}(0) + \boldsymbol{\delta}_y \boldsymbol{\delta}_x \frac{\partial}{\partial y}(bx) + \boldsymbol{\delta}_y \boldsymbol{\delta}_y \frac{\partial}{\partial y}(0) + \boldsymbol{\delta}_y \boldsymbol{\delta}_z \frac{\partial}{\partial y}(0) \\
&\quad + \boldsymbol{\delta}_z \boldsymbol{\delta}_x \frac{\partial}{\partial z}(bx) + \boldsymbol{\delta}_z \boldsymbol{\delta}_y \frac{\partial}{\partial z}(0) + \boldsymbol{\delta}_z \boldsymbol{\delta}_z \frac{\partial}{\partial z}(0) \\
&= b \boldsymbol{\delta}_x \boldsymbol{\delta}_x \\
\nabla \cdot \mathbf{v}\mathbf{v} &= \boldsymbol{\delta}_x \left[\frac{\partial}{\partial x}(b^2 x^2) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(0)\right] + \boldsymbol{\delta}_y \left[\frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(0)\right] \\
&\quad + \boldsymbol{\delta}_z \left[\frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(0)\right] \\
&= 2b^2 x \boldsymbol{\delta}_x.
\end{aligned}$$

Part (c)

The third vector field has components $v_x = by$, $v_y = bx$, and $v_z = 0$. Using the formulas, we have

$$\begin{aligned}
\nabla \cdot \mathbf{v} &= \frac{\partial}{\partial x}(by) + \frac{\partial}{\partial y}(bx) + \frac{\partial}{\partial z}(0) = 0 \\
\nabla \mathbf{v} &= \boldsymbol{\delta}_x\boldsymbol{\delta}_x\frac{\partial}{\partial x}(by) + \boldsymbol{\delta}_x\boldsymbol{\delta}_y\frac{\partial}{\partial x}(bx) + \boldsymbol{\delta}_x\boldsymbol{\delta}_z\frac{\partial}{\partial x}(0) + \boldsymbol{\delta}_y\boldsymbol{\delta}_x\frac{\partial}{\partial y}(by) + \boldsymbol{\delta}_y\boldsymbol{\delta}_y\frac{\partial}{\partial y}(bx) + \boldsymbol{\delta}_y\boldsymbol{\delta}_z\frac{\partial}{\partial y}(0) \\
&\qquad + \boldsymbol{\delta}_z\boldsymbol{\delta}_x\frac{\partial}{\partial z}(by) + \boldsymbol{\delta}_z\boldsymbol{\delta}_y\frac{\partial}{\partial z}(bx) + \boldsymbol{\delta}_z\boldsymbol{\delta}_z\frac{\partial}{\partial z}(0) \\
&= b\boldsymbol{\delta}_x\boldsymbol{\delta}_y + b\boldsymbol{\delta}_y\boldsymbol{\delta}_x \\
\nabla \cdot \mathbf{vv} &= \boldsymbol{\delta}_x\left[\frac{\partial}{\partial x}(b^2y^2) + \frac{\partial}{\partial y}(b^2xy) + \frac{\partial}{\partial z}(0)\right] + \boldsymbol{\delta}_y\left[\frac{\partial}{\partial x}(b^2xy) + \frac{\partial}{\partial y}(b^2x^2) + \frac{\partial}{\partial z}(0)\right] \\
&\qquad + \boldsymbol{\delta}_z\left[\frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(0)\right] \\
&= b^2x\boldsymbol{\delta}_x + b^2y\boldsymbol{\delta}_y.
\end{aligned}$$

Part (d)

The fourth vector field has components $v_x = -by$, $v_y = bx$, and $v_z = 0$. Using the formulas, we have

$$\begin{aligned}
\nabla \cdot \mathbf{v} &= \frac{\partial}{\partial x}(-by) + \frac{\partial}{\partial y}(bx) + \frac{\partial}{\partial z}(0) = 0 \\
\nabla \mathbf{v} &= \boldsymbol{\delta}_x\boldsymbol{\delta}_x\frac{\partial}{\partial x}(-by) + \boldsymbol{\delta}_x\boldsymbol{\delta}_y\frac{\partial}{\partial x}(bx) + \boldsymbol{\delta}_x\boldsymbol{\delta}_z\frac{\partial}{\partial x}(0) + \boldsymbol{\delta}_y\boldsymbol{\delta}_x\frac{\partial}{\partial y}(-by) + \boldsymbol{\delta}_y\boldsymbol{\delta}_y\frac{\partial}{\partial y}(bx) + \boldsymbol{\delta}_y\boldsymbol{\delta}_z\frac{\partial}{\partial y}(0) \\
&\qquad + \boldsymbol{\delta}_z\boldsymbol{\delta}_x\frac{\partial}{\partial z}(-by) + \boldsymbol{\delta}_z\boldsymbol{\delta}_y\frac{\partial}{\partial z}(bx) + \boldsymbol{\delta}_z\boldsymbol{\delta}_z\frac{\partial}{\partial z}(0) \\
&= b\boldsymbol{\delta}_x\boldsymbol{\delta}_y - b\boldsymbol{\delta}_y\boldsymbol{\delta}_x \\
\nabla \cdot \mathbf{vv} &= \boldsymbol{\delta}_x\left[\frac{\partial}{\partial x}(b^2y^2) + \frac{\partial}{\partial y}(-b^2xy) + \frac{\partial}{\partial z}(0)\right] + \boldsymbol{\delta}_y\left[\frac{\partial}{\partial x}(-b^2xy) + \frac{\partial}{\partial y}(b^2x^2) + \frac{\partial}{\partial z}(0)\right] \\
&\qquad + \boldsymbol{\delta}_z\left[\frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(0)\right] \\
&= -b^2x\boldsymbol{\delta}_x - b^2y\boldsymbol{\delta}_y.
\end{aligned}$$